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M.Sc. (Sem.-I) Examination November - 2025

Mathematics : Special Functions-I (PGMTH : 1045)

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions (1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) Answer all questions. (3) Figure to the right indicates marks of the questions. (4) Follow usual notations and conventions.	Seat No.: Student's Signature

Q.1. Answer any <u>TWO</u> of the following **14**

(I) Define uniform convergence of the infinite product. If for a positive constant M_n such that $\sum_{n=1}^{\alpha} M_n$ is convergent and $|a_n(z)| < M_n$ for all z in the complex region R, show that the product $\prod_{n=1}^{\infty}(1+a_n)$ is uniformly convergent in R.

(II) Define absolute convergence of an infinite product. Show that the product $\prod_{n=1}^{\infty}(1+a_n)$ with zero factor deleted is absolutely convergent iff $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(III) Discuss the convergence of the product $\prod_{n=1}^{\infty} \exp\left(\frac{1}{n}\right)$ and $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

Q.2. Answer any <u>TWO</u> of the following **14**

(I) If $a+b+\frac{1}{2}$ is neither zero nor a negative integer and if $|x| < 1$, $|4x(1-x)| < 1$ Prove that

$$F\left(a, b; a+b+\frac{1}{2}; 4x(1-x)\right) = F\left(2a, 2b; a+b+\frac{1}{2}; x\right)$$

(II) Define Hypergeometric functions. Show that the series given in the Hypergeometric function is absolutely convergent for $|z| = 1$ if $\mathrm{Re}(c-a-b) > 0$

(III) With usual notation prove that $F\begin{bmatrix} a, b; & \\ & -1 \\ c; & \end{bmatrix} = \dfrac{\Gamma(1+a-b)\,\Gamma\left(1+\frac{a}{2}\right)}{\Gamma(1+a)\,\Gamma\left(1+\frac{a}{2}-b\right)}$

Q.3. Answer any <u>TWO</u> of the following **14**

(I) Define contiguous function to $F(a, b; c; z)$ and derive the relations.

(i) $(a-c+1)F = aF(a+) - (c-1)F(c-)$

(ii) $[a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+)$

(II) If 2b is either zero nor a negative integral and if $|y| < 1/2$ and $|y/(1 - y)| < 1$ then show that

$$(1-y)^{-a} F\begin{bmatrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2}; & \\ & \frac{y^2}{(1-y)^2} \\ b+\frac{1}{2}; & \end{bmatrix} = F\begin{bmatrix} a, b; & \\ & 2y \\ 2b; & \end{bmatrix}$$

(III) Derive the equation $z(1-z)w'' + [c-(a+b+1)z]\,w' - abw = 0$.

Q.4. Answer any <u>TWO</u> of the following **14**

(I) Derive the values $\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$

(II) Define Factorial function. Prove that

(a) $(\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ **(b)** $(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n$

(III) In usual notation prove that $B(p, q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$; p q>0

Q.5. Answer any <u>TWO</u> of the following **14**

(I) If n is an integer and Re(Z) > 0, show that $\Gamma(z) = \lim\limits_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$

(II) If Z > 0 and γ be Euler constant then in usual notation prove that

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(z+1)}$$

(III) Obtain the values of $\Gamma(1)$ *and* $\Gamma'(1)$
